Clifford involutions following Hitzer and Sangwine

Robin K. S. Hankin

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Clifford inverses

Hitzer and Sangwine set up a number of involutions which I reproduce for convenience below. Given $M \in Cl(p,q)$ and

$$M = \langle M\rangle_0 + \langle M\rangle_1 + \langle M\rangle_2 + \cdots + \langle M\rangle_0$$

we have a number of involutions, documented at `involutions.Rd`:

- The main grade involution $\widehat{M} = \sum_{k=0}^{n}(-1)^k \langle M\rangle_k$ `gradeinv(M)`
- Reversion $\widetilde{M} = \sum_{k=0}^{n}(-1)^{k(k-1)/2} \langle M\rangle_k$ `rev(M)`
- Clifford conjugation $\overline{M} = \sum_{k=0}^{n}(-1)^{k(k+1)/2} \langle M\rangle_k$ `cliffconj(M)`
- Grade specific maps $m_{\overline{j},\overline{k}}(M) = M - 2\left(\langle M\rangle_j + \langle M\rangle_k\right)$ `neg(M)`
- The generalised grade specific map $m_A(M) = M - \sum_{i\in A}\langle M\rangle i$ `neg(M)`

H&S assert that

$$\overline{M} = \widehat{\widetilde{M}} = \widetilde{\widehat{M}} = \sum_{k=0}^{n}(-1)^{k(k+1)/2} \langle M\rangle_k$$

which we may verify numerically:

```
library(clifford)
```

```
(M <- rcliff())
```

```
## Element of a Clifford algebra, equal to
## + 4 + 5e_1 + 4e_2 - 2e_13 - 3e_23 - 1e_123 + 3e_4 + 1e_145 + 2e_1456
```

```
a1 <- cliffconj(M)
a2 <- gradeinv(rev(M))
a3 <- rev(gradeinv(M))
is.zero(a2-a1) & is.zero(a3-a1)
```

```
## [1] TRUE
```

$p+q=3$, three-dimensional vector spaces.

We now consider the case $p+q=3$. If $x \in Cl(p,q)$ with $p+q=3$ then equation 6.2 asserts that $x\overline{x} = r_0 + r_3e_1e_2e_3$ for some $r_0, r_3 \in \mathbb{R}$:

```
(x <- rcliff(d=3,g=3))
```

```
## Element of a Clifford algebra, equal to
## + 3 - 3e_1 - 1e_12 + 3e_23 + 1e_123
```

```
x*cliffconj(x)
```

```
## Element of a Clifford algebra, equal to
## + 9 + 24e_123
```

and equation 6.3 asserts that $x\overline{x}(x\overline{x})^{\sim} \in \mathbb{R}$:

```
f <- function(x){
    jj <- x*cliffconj(x)
    is.real(jj*rev(jj))
}
```

```
signature(0,3)
f(rcliff(d=3,g=3))
```

```
## [1] TRUE
```

```
signature(1,2)
f(rcliff(d=3,g=3))
```

```
## [1] TRUE
```

```
signature(2,1)
f(rcliff(d=3,g=3))
```

```
## [1] TRUE
```

```
signature(3,0)
f(rcliff(d=3,g=3))
```

```
## [1] TRUE
```

Thus equation 6.5, which asserts that the right inverse x_r^{-1} is

$$x_r^{-1} = \frac{\overline{x}\hat{x}\tilde{x}}{x\overline{x}\hat{x}\tilde{x}}, \qquad xx_r^{-1} = 1$$

```
RI3 <- function(x){ # right inverse
    jj <- cliffconj(x)*gradeinv(x)*rev(x)
    return(jj/drop(x*jj))
}
```

```
a <- 5+rcliff(d=3,g=3)
a
```

```
## Element of a Clifford algebra, equal to
## + 9 + 5e_1 + 1e_2 + 3e_23 - 3e_123
```

```
RI3(a)
```

```
## Element of a Clifford algebra, equal to
## + 0.07409979 - 0.00228152e_1 - 0.005455808e_2 + 0.008332507e_13 -
```

```
## 0.05802996e_23 + 0.05862514e_123
```

```r
zap(a*RI3(a))
```

```
## [1] 1
```

```r
zap(RI3(a)*a)
```

```
## [1] 1
```

Now equations 7.7 and 7.8, which assert that if $x\overline{x}m_{\overline{3},\overline{4}}(x\overline{x})$ is nonzero, we have

$$x_r^{-1} = \frac{\overline{x}m_{\overline{3},\overline{4}}(x\overline{x})}{x\overline{x}m_{\overline{3},\overline{4}}(x\overline{x})}, \qquad xx_r^{-1} = 1$$

and

$$x_l^{-1} = \frac{\overline{x}m_{\overline{3},\overline{4}}(x\overline{x})}{\overline{x}m_{\overline{3},\overline{4}}(x\overline{x})x}, \qquad x_l^{-1}x = 1$$

Numerical verification:

```r
f77 <- function(x){
    jj <- cliffconj(x)*neg(x*cliffconj(x),3:4)
    return(jj/drop(x*jj))
}

f78 <- function(x){
    jj <- neg(cliffconj(x)*x,3:4)*cliffconj(x)
    return(jj/drop(jj*x))
}

a <- 3 + rcliff(d=4)
a
```

```
## Element of a Clifford algebra, equal to
## + 7 - 3e_2 + 2e_23 + 1e_123 + 5e_4 - 2e_134 + 3e_1234
```

```r
f77(a)
```

```
## Element of a Clifford algebra, equal to
## + 0.1540342 - 0.01369193e_1 + 0.06308068e_2 + 0.0205379e_13 - 0.04205379e_23 -
## 0.02689487e_123 - 0.09633252e_4 - 0.04107579e_14 - 0.008801956e_24 +
## 0.01760391e_124 + 0.02542787e_34 + 0.02933985e_134 + 0.06845966e_234 -
## 0.05427873e_1234
```

```r
zap(a*f77(a))
```

```
## [1] 1
```

```r
zap(f77(a)*a)
```

```
## [1] 1
```

Try the different signatures:

```r
set.seed(0)
sigs <- 0:4
left <- rep(NA,5)
```

```
right <- rep(NA,5)
diff <- rep(NA,5)
for(i in seq_along(sigs)){
    signature(sigs[i])
    a <- sample(1:9,1) + rcliff(d=4)
    left[i]  <- Mod(a*f77(a) -1)
    right[i] <- Mod(f77(a)*a -1)
    diff[i] <- Mod(f77(a)-f78(a))
}
left
```

```
## [1] 0.000000e+00 0.000000e+00 4.336809e-18 2.775558e-17 3.156220e-17
```

```
right
```

```
## [1] 0.000000e+00 0.000000e+00 4.336809e-18 2.775558e-17 3.151450e-17
```

```
diff
```

```
## [1] 0 0 0 0 0
```

Just to be explicit, the following DOES NOT WORK:

```
a <- rcliff()
a*f77(a)   # (denominator not real)
```

The case $p + q \leq 5$

Right inverse

Equation 8.21 asserts that, if $p + q \leq 5$ then $z = x\overline{x}\hat{x}\tilde{x}m_{\overline{1},\overline{4}}(x\overline{x}\hat{x}\tilde{x}) \in \mathbb{R}$.
Equation 8.22 asserts that, if z is nonzero, then

$$x_r^{-1} = \frac{\overline{x}\hat{x}\tilde{x}m_{\overline{1},\overline{4}}(x\overline{x}\hat{x}\tilde{x})}{x\overline{x}\hat{x}\tilde{x}m_{\overline{1},\overline{4}}(x\overline{x}\hat{x}\tilde{x})}, xx_r^{-1} = 1.$$

```
f822 <- function(x){
    jj <- cliffconj(x)*gradeinv(x)*rev(x)
    jj <- jj*neg(x*jj,c(1L,4L))
    jj/drop(zap(x*jj))
}
```

```
a <- 7+clifford(list(1,3,5,1:2,c(1,5),c(3,4),1:3,2:4,c(2,3,5),1:4,2:5,c(1,2,3,5),1:5),1:13)
a
```

```
## Element of a Clifford algebra, equal to
## + 7 + 1e_1 + 4e_12 + 2e_3 + 7e_123 + 6e_34 + 8e_234 + 10e_1234 + 3e_5 + 5e_15 +
## 9e_235 + 12e_1235 + 11e_2345 + 13e_12345
```

```
f822(a)
```

```
## Element of a Clifford algebra, equal to
## + 0.04709949 + 0.002540291e_1 - 0.05463707e_2 - 0.05130554e_12 - 0.0388956e_3 -
## 0.01465873e_13 + 0.003414817e_23 - 0.01070254e_123 + 0.00466414e_4 +
## 0.006829634e_14 - 0.02798484e_24 + 0.001332612e_124 - 0.04938991e_34 +
## 0.006829634e_134 - 0.01499188e_234 + 0.004914005e_1234 - 0.0588067e_5 -
## 0.01823902e_15 - 0.09401385e_25 - 0.05101002e_125 - 0.02640802e_35 +
## 0.05150748e_135 - 0.09563797e_235 - 0.07498127e_1235 + 0.02165381e_45 -
```

```
## 0.01360835e_145 + 0.04395504e_245 + 0.07644131e_1245 - 0.04065396e_345 -
## 0.06661237e_1345 + 0.01847141e_2345 - 0.02107641e_12345
```

```
zap(a*f822(a))
```

```
## [1] 1
```

```
zap(f822(a)*a)
```

```
## [1] 1
```

And a similar set of verifications:

```
sigs <- 0:6
diffl <- rep(NA,5)
diffr <- rep(NA,5)
for(i in seq_along(sigs)){
    signature(sigs[i])
    a <- sample(1:9,1) + rcliff(d=5)
    diffl[i] <- Mod(a*f822(a)-1)
    diffr[i] <- Mod(f822(a)*a-1)
}
diffl
```

```
## [1] 0.000000e+00 3.469447e-18 3.469447e-18 3.078242e-16 3.974751e-18
## [6] 4.958112e-17 7.933703e-17
```

```
diffr
```

```
## [1] 0.000000e+00 3.469447e-18 3.469447e-18 3.078242e-16 3.974751e-18
## [6] 4.958112e-17 8.283200e-17
```

Left inverse

Similarly, equation 8.23 asserts that, if $p+q\leq 5$ then $z'=m_{\overline{1},\overline{4}}(\tilde{x}\hat{x}\overline{x}x)\tilde{x}\hat{x}\overline{x}x\in\mathbb{R}$. And if $z'\neq 0$ equation 8.24 asserts that

$$x_l^{-1}=\frac{m_{\overline{1},\overline{4}}(\tilde{x}\hat{x}\overline{x}x)\tilde{x}\hat{x}\overline{x}}{m_{\overline{1},\overline{4}}(\tilde{x}\hat{x}\overline{x}x)\tilde{x}\hat{x}\overline{x}x},\qquad x_l^{-1}x=1.$$

The R idiom would be

```
f824 <- function(x){  # left inverse
    jj <- rev(x)*gradeinv(x)*cliffconj(x)
    jj <- neg(jj*x,c(1L,4L))*jj
    jj/drop(zap(jj*x))
}
```

Check:

```
zap(f824(x)*x)
```

```
## [1] 1
```

```
zap(f822(x)*x)
```

```
## [1] 1
```

It turns out that the left and right inverses coincide:

```
signature(0,5)
Mod(f822(x) - f824(x))
```

```
## [1] 0
```

```
signature(1,4)
Mod(f822(x) - f824(x))
```

```
## [1] 0
```

```
signature(2,3)
Mod(f822(x) - f824(x))
```

```
## [1] 0
```

```
signature(3,2)
Mod(f822(x) - f824(x))
```

```
## [1] 0
```

```
signature(4,1)
Mod(f822(x) - f824(x))
```

```
## [1] 0
```

Cartan isomorphism

We will carry out Cartan's isomorphism from $Cl(p,q)$ to $Cl(p-4,q+4)$ numerically. Here we specify $p+q=7$ by calling `rcliff()` with argument `d=7`, and force $p=4$ by executing `signature(4)`:

```
a <- rcliff(d=7)    # Cl(4,3)
b <- rcliff(d=7)    # Cl(4,3)
signature(4,3)      # e1^2 = e2^2 = e3^2 = e4^2 = +1; e5^2 = ... = -1
ab <- a*b           # multiplication in Cl(4,3)

signature(0,7)   # e1^2 = ... = -1
cartan(a)*cartan(b) == cartan(ab) # multiplication in Cl(0,7)
```

```
## [1] TRUE
```

and again using `cartan_inverse()`:

```
cartan_inverse(cartan(a) * cartan(b)) == ab  # precalculated product!
```

```
## [1] TRUE
```

Now try mapping $Cl(5,2) \longrightarrow Cl(1,7)$:

```
signature(5,2); ab_sig5 <- a*b

signature(1,7)
cartan(a,2) * cartan(b,2)  == cartan(ab_sig5,2)
```

```
## [1] TRUE
```

```
cartan_inverse(cartan(a,2) * cartan(b,2),2)  == ab_sig5
```

```
## [1] TRUE
```